HOW TO SOLVE IT?

Transitioning to university level courses

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Objectives:

- To introduce flowcharts to organize steps in problem-solving and note-taking.
- To introduce Polya's 4-step problem-solving model

Polya's 4-step problem-solving model

First Principle: Understand the problem

- What are you asked to find or show?
- Can you restate the problem in your own words?
- Can you think of a picture or a diagram to visualize or help you understand the problem?
- Is there enough information? Is all the information relevant to find a solution?
- Do you understand all the words used in stating the problem?
- Do you need to ask a question to get the answer?

Polya’s 4-step problem-solving model

Second Principle: Devise a plan

- Guess and Check
- Make an orderly list
- Eliminate possibilities
- Use symmetry
- Consider special cases
- Use direct reasoning
- Solve an equation
- Look for a pattern
- Draw a picture
- Solve a simpler problem
- Work backward
- Use a formula

Polya's 4-step problem-solving model

Third Principle: Carry out the plan

- Be patient and thorough.
- Persist with the chosen plan, if it does not work, discard it and choose another one.
- Be creative

Fourth Principle: Review and Extend

- Does the answer make sense?
- Review what worked and what did not work
- Can you extend the result?

Understand the problem

- Write down or draw to visualize the problem
- Understand all words, terms, notations, and symbols.
- Look for other related information.

Devise a Plan

- Understand the goal. What are the known and unknown?
- Look at other problems or examples for inspiration
- Choose a smaller part to try or work on a simpler case

Carry out the Plan

- Focus on one part at a time. Work on a simpler case
- Regularly check with the goal. Keep track of the goal.
- Try something else if it is not working

Check and Interpret Answer

- Have we reached the goal? Does the answer make sense?
- Rewrite to summarize thought process clearly to others
- Find something we can learn.

Credit: David Butler & Nicholas Crouch 2020 The University of Adelaide. Inspired by "How to Solve it" by G. Polya

Concept Map of Algebra

*Concept map is a variation of the flowchart when we mainly demonstrate the interrelationships (not a flow).

Solving Equations
- Solving linear equations
- Solving quadratic equations
- Solving polynomial equations
- Solving rational equations
- Solving exponential equations
- Solving logarithmic equations
- Solving trigonometric equations

Understanding Functions and their Graphs
- Linear functions
- Quadratic functions
- Polynomials
- Rational functions
- Exponential functions
- Logarithmic functions
- Trigonometric functions

Solving Inequalities
- Solving linear inequalities
- Solving quadratic inequalities
- Solving polynomial inequalities
- Solving rational inequalities
- Solving exponential inequalities
- Solving logarithmic inequalities
- Solving trigonometric inequalities

Supporting Algebra Techniques
- Factorization
- Finding least common denominator
- Completing squares
- Polynomial division
- Partial fraction decomposition->Calculus
- Laws of exponents and logarithms
- Trigonometric identities
- Compose and decompose functions
- Take inverse of a function

Conic Sections and their Graphs (Non-functions mainly)
- Parabolas
- Ellipses (Circles included)
- Hyperbolas

Utilizing the flowchart in problem solving (MATH)

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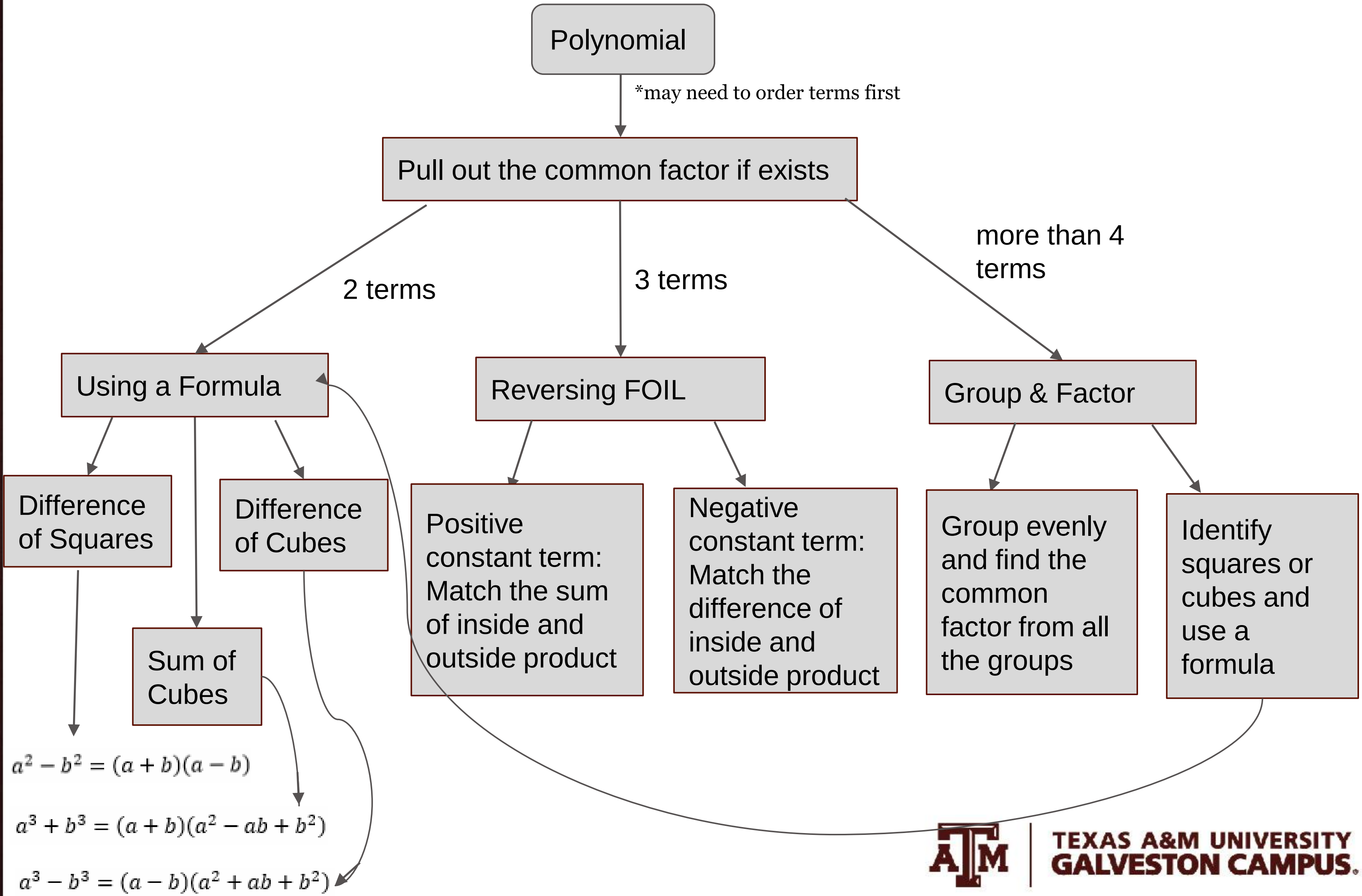
Flowchart of Factoring Polynomials



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Question 1:

Factor the polynomial completely over the real numbers

$$3x^2 - 9x + 6$$

Question 1: **SOLUTION PROCESS**

Factor the polynomial *completely* over the *real numbers*

$$3x^2 - 9x + 6$$

Step 1: Understanding the problem

Common mistake in this type of problem is to set the polynomial expression to zero and solve for x.

- To *factor* means to write as a *product* of terms
- To factor *completely* means the final expression cannot be factored anymore
- To factor over the *real numbers* means all numbers involved must be real (can be fractions, decimals, whole numbers, radicals, positive, negative, or zero)

Step 2: Devise a plan

Using the flowchart, we see that we first need to pull out a common factor.

Step 3: Carry out the plan

$$\begin{aligned} 3x^2 - 9x + 6 &= 3(x^2 - 3x + 2) \\ &= 3(x-1)(x-2) \end{aligned}$$

Step 4: Check

Using distributive property or FOIL(multiplying all terms),
we verify that $3(x-1)(x-2) = 3x^2 - 9x + 6$

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Question 1: WRITE-UP OF SOLUTION

Factor the polynomial completely over the real numbers

$$3x^2 - 9x + 6$$

Solution:

$$\begin{aligned} 3x^2 - 9x + 6 &= 3(x^2 - 3x + 2) \\ &= 3(x-1)(x-2) \end{aligned}$$

Answer: $\mathbf{3(x-1)(x-2)}$

Note: Rewriting the process from previous page more succinctly as shown above is a sufficient show of solution for this problem.

You will be expected to show your work thoroughly in your math courses.

Question 2:

Factor the polynomial completely over the real numbers

$$x^2 - 3$$

Question 2: **SOLUTION PROCESS**

Factor the polynomial *completely* over the *real numbers*

$$x^2 - 3$$

Step 1: Understanding the problem

Common mistake in this type of problem is to say that there is no answer or that the given expression cannot be factored. This is where reading the problem carefully is crucial!

- To factor over the *real numbers* means all numbers involved must be real (can be fractions, decimals, whole numbers, radicals, positive, negative, or zero). There are real numbers whose square is 3, those numbers are $\pm\sqrt{3}$

Step 2: Devise a plan

Using the flowchart, we use the difference of square formula. Since 3 is not a perfect square, we think about a real number whose square is 3.

Step 3: Carry out the plan

$$x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$$

Step 4: Check

Using distributive property or FOIL(multiplying all terms),

we verify that $(x - \sqrt{3})(x + \sqrt{3}) = x^2 - 3$

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Question 2: **WRITE-UP OF SOLUTION**

Factor the polynomial completely over the real numbers

$$x^2 - 3$$

Solution:

$$x^2 - 3 = (x - \sqrt{\mathbf{3}})(x + \sqrt{\mathbf{3}})$$

Answer: $(\boldsymbol{x} - \sqrt{\mathbf{3}})(\boldsymbol{x} + \sqrt{\mathbf{3}})$

REMARK: If the problem had been *"Factor the polynomial completely over the integers",* then we cannot factor $x^2 - 3$ since $\sqrt{3}$ is not an integer(a whole number)

So, be mindful about the type of numbers over which you need to factor your polynomial. Read your problem carefully!

Question 3:

Factor the polynomial completely over the real numbers

$$x^2 - 9 + y^2 - 2xy$$

Question 3: **SOLUTION PROCESS**

Factor the polynomial *completely* over the *real numbers*

$$x^2 - 9 + y^2 - 2xy$$

Step 1: Understanding the problem

We have a polynomial in two variables.

To *factor* means to write as a *product* of terms

Step 2: Devise a plan

Using the flowchart, we will try grouping

Step 3: Carry out the plan

$$x^2 - 9 + y^2 - 2xy = (x^2 - 9) + (y^2 - 2xy)$$
$$= (x - 3)(x + 3) + y(y - 2x)$$

While we can factor $(x^2 - 9)$ and $(y^2 - 2xy)$ separately, the final expression is still a sum not a product. So, we try another way of grouping.

Rewriting expression: $x^2 - 9 + y^2 - 2xy = x^2 - 2xy + y^2 - 9$

Grouping: $= x^2 - 2xy + y^2 \; - 9$

$= (x - y)^2 \; - \; 9$

**The last expression looks like difference of squares $a^2 - b^2$

$= (x - y)^2 \; -3^2$

$= ((x - y) \; -3)\,((x - y) \; + 3)$

The last expression is now a product of terms, so we're done.

Answer: $(x - y - 3)(x - y + 3)$

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Question 3: WRITE-UP OF SOLUTION

Factor the polynomial completely over the real numbers

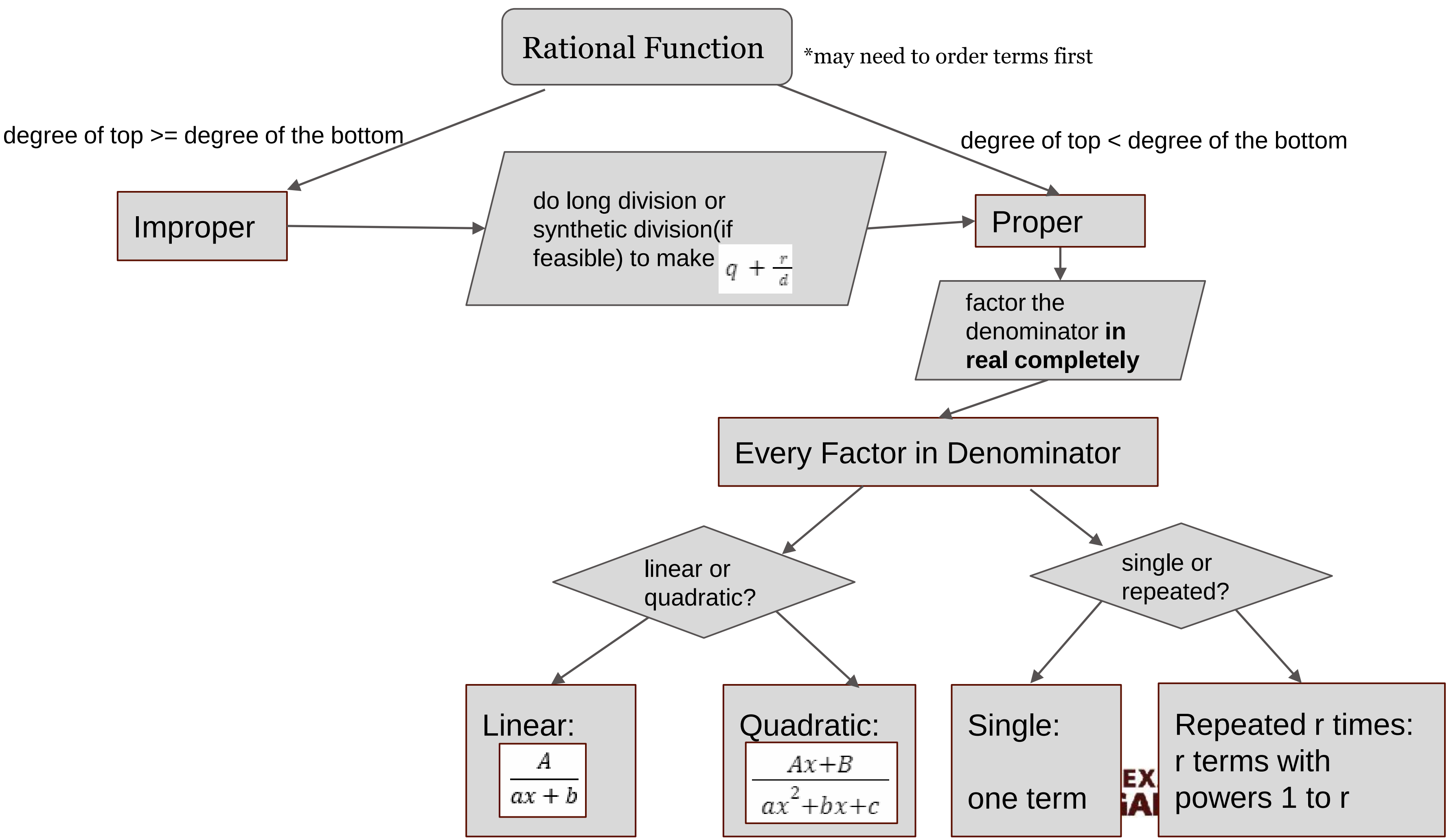
$$x^2 - 9 + y^2 - 2xy$$

SOLUTION:

$$\begin{aligned} x^2 - 9 + y^2 - 2xy &= x^2 - 2xy + y^2 - 9 \\ &= x^2 - 2xy + y^2 - 9 \\ &= (x-y)^2 - 9 \\ &= (x-y)^2 - 3^2 \\ &= ((x-y) - 3)((x-y) + 3) \\ &= (x-y-3)(x-y+3) \end{aligned}$$

ANSWER: $(\boldsymbol{x} - \boldsymbol{y} - \mathbf{3})(\boldsymbol{x} - \boldsymbol{y} + \mathbf{3})$

Flowchart of Decomposing Rational Functions (Partial Fractions Decomposition)



Question 4:

Select all improper rational expressions

a) $\frac{x-1}{x^3+x}$

b) $\frac{5x^3+2x+4}{(x-2)(x+3)}$

c) $\frac{x-4}{x+6}$

d) $\frac{5}{x^3-1}$

Question 4: Select all improper rational expressions

By Definition, if the degree of the top expression is greater than or equal to the degree of the bottom expression, then the rational is improper

Note: it's a good practice to make sure we do not have division by zero when working with rationals, i.e. make sure we exclude all real x-values that make the denominator(bottom expression) zero

a) $\frac{x-1}{x^3+x}$

Degree of the numerator(top): 1
Degree of the denominator(bottom): 3
Rational is proper
**Note: $x = 0$ will lead to division by zero so the rational is defined for all real x values but 0!

b) $\frac{5x^3+2x+4}{(x-2)(x+3)}$

Degree of the numerator(top): 3
Degree of the denominator(bottom): 2
Rational is improper
**Note: $x = -3$, $x = 2$ will lead to division by zero so the rational is defined for all real x values but -3 and 2!

c) $\frac{x-4}{x+6}$

Degree of the numerator(top): 1
Degree of the denominator(bottom): 1
Rational is improper
**Note: $x = -6$ will lead to division by zero so the rational is defined for all real x values but -6!

d) $\frac{5}{x^3-1}$

Degree of the numerator(top): 0 **Note: $5 = 5 \cdot x^0$. Recall: $x^0 = 1$ for all *non-zero* real x.
Degree of the denominator(bottom): 3=1
Rational is proper
**Note: $x = 1$ will lead to division by zero so the rational is defined for all real x values but 1!

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Question 5:

If a proper rational expression has a denominator of $x^2(x-4)(x^2+3)$, select the correct form of the partial fractions

- $\frac{A}{x^2} + \frac{B}{x-4} + \frac{Cx+D}{x^2+3}$
- $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4} + \frac{D}{x^2+3}$
- $\frac{A}{x^2} + \frac{B}{x-4} + \frac{Cx+D}{x^2+3}$
- $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4} + \frac{Dx+E}{x^2+3}$

Question 5: **SOLUTION**

If a *proper* rational expression has a *denominator* of $x^2(x-4)(x^2+3)$, select the correct form of the partial fractions

repeat

Factors of the denominator	Type	Exponent of the factor	Partial Fractions
x	Linear	2	$\frac{A}{x}+\frac{B}{x^2}$
$(x-4)$	Linear	1	$\frac{C}{x-4}$
(x^2+3)	Quadratic	1	$\frac{Dx+E}{x^2+3}$

- ○ $\frac{A}{x^2}+\frac{B}{x-4}+\frac{Cx+D}{x^2+3}$
- ○ $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-4}+\frac{D}{x^2+3}$
- ○ $\frac{A}{x^2}+\frac{B}{x-4}+\frac{Cx+D}{x^2+3}$
- ○ $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-4}+\frac{Dx+E}{x^2+3}$

Question 6:

If a proper rational expression has a denominator of $(2x + 5)(x^2 + 3x + 5)^2$, select the correct form of the partial fractions

- $\frac{A}{2x+5} + \frac{Bx+C}{x^2+3x+5}$
- $\frac{A}{2x+5} + \frac{Bx+C}{(x^2+3x+5)^2}$
- $\frac{A}{2x+5} + \frac{Bx+C}{x^2+3x+5} + \frac{Dx+E}{(x^2+3x+5)^2}$
- $\frac{A}{2x+5} + \frac{B}{x^2+3x+5} + \frac{C}{(x^2+3x+5)^2}$

Question 6: **SOLUTION**

exponent repeat

If a proper rational expression has a denominator of $(2x+5)(x^2+3x+5)^2$, select the correct form of the partial fractions

Factors of the denominator	Type	Exponent of the factor	Partial Fractions
$(2x+5)$	Linear	1	$\frac{A}{2x+5}$
(x^2+3x+5)	Quadratic	2	$\frac{Bx+C}{x^2+3x+5}+\frac{Dx+E}{(x^2+3x+5)^2}$

- $\frac{A}{2x+5}+\frac{Bx+C}{x^2+3x+5}$
- $\frac{A}{2x+5}+\frac{Bx+C}{(x^2+3x+5)^2}$
- $\frac{A}{2x+5}+\frac{Bx+C}{x^2+3x+5}+\frac{Dx+E}{(x^2+3x+5)^2}$
- $\frac{A}{2x+5}+\frac{B}{x^2+3x+5}+\frac{C}{(x^2+3x+5)^2}$

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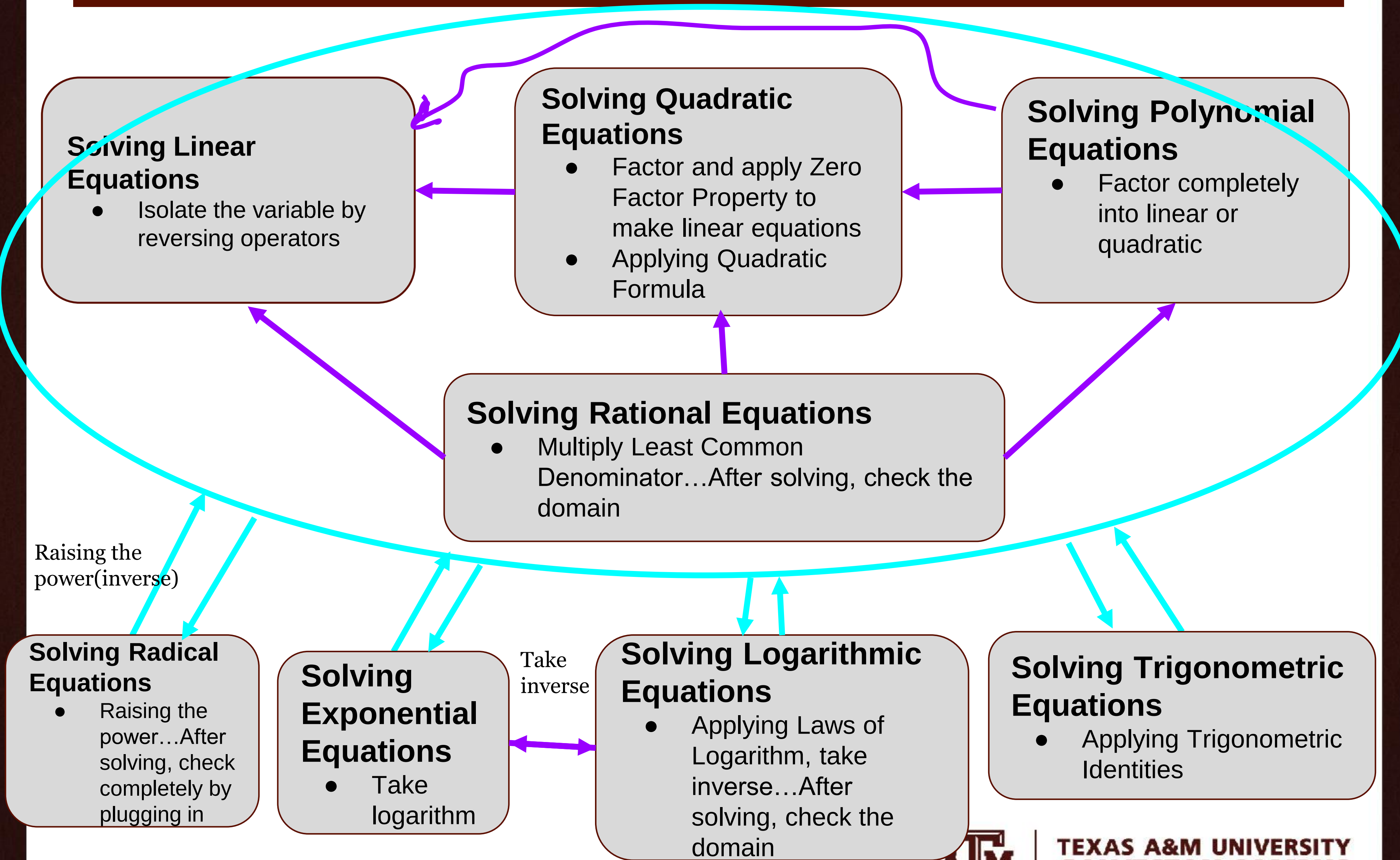
Flowchart of Solving Equations



*Composition of Functions: **Solve from outside to inside**

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Question 7:

Does the following function ever cross or touch the x-axis?

$$R(m) = \frac{5m - 4}{m^2 - m - 2}$$

Question 7: SOLUTION PROCESS – using Polya's 4-step model

Does the following function ever cross or touch the x-axis?

$$R(m) = \frac{5m - 4}{m^2 - m - 2}$$

Step 1: Understand the problem

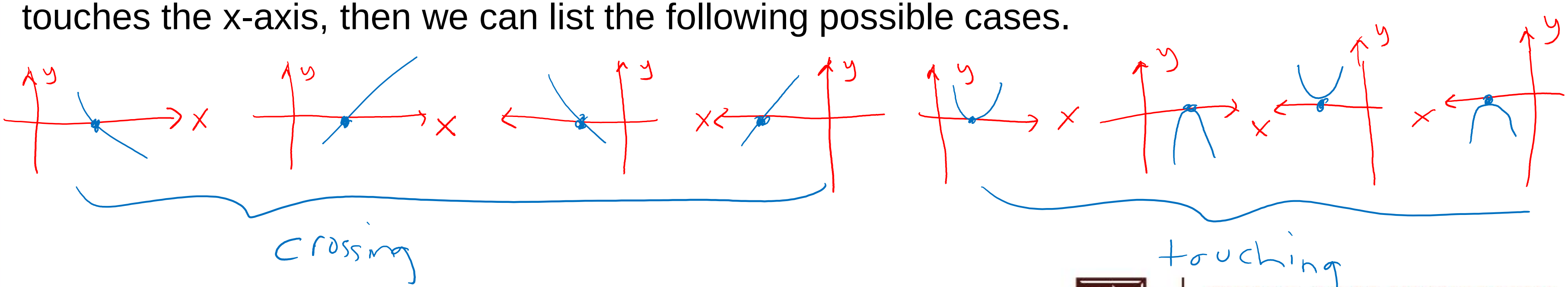
Without looking at a graphing calculator, we want to check whether the function touches or crosses the x-axis.

- m is the independent variable, i.e. it's the x-variable
- $R(m)$is the dependent variable, i.e. it's the y-variable

Step 2: Devise a plan

Since we have a rational function, we cannot have a division by zero.
$m^2 - m - 2 = (m-2)(m+1) = 0 \rightarrow m = 2, -1$ must be excluded to avoid zero in denominator. Furthermore, this implies that at $m = 2, -1$, the graph does not touch the x-axis

Even if we don't know what the graph of the function looks like, if the function crosses or touches the x-axis, then we can list the following possible cases.



What do these figures have in common?

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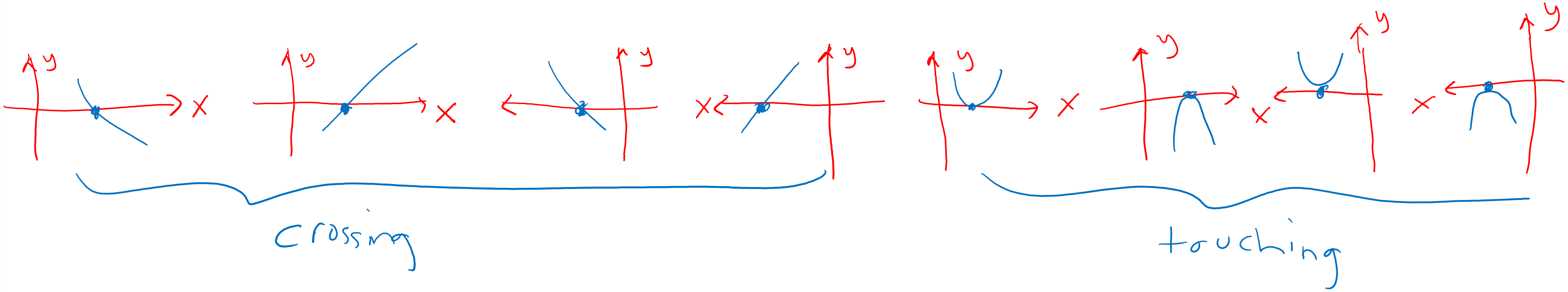
Question 7: SOLUTION PROCESS – using Polya's 4-step model

Does the following function ever cross or touch the x-axis?

$$R(m) = \frac{5m - 4}{m^2 - m - 2}$$

Step 3: Carry out the plan

If the function crosses or touches the x-axis, then in each of these figures, we see that the y-component is zero. So, this means that if we set $y = 0$ then we can find where the function at least touches the x-axis.



$$\frac{5m - 4}{m^2 - m - 2} = 0$$

$$\frac{5m - 4}{m^2 - m - 2} \cdot (m^2 - m - 2) = 0 \cdot (m^2 - m - 2)$$

$$5m - 4 = 0 \rightarrow m = \frac{4}{5}$$

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Question 7: SOLUTION PROCESS – using Polya's 4-step model

Does the following function ever cross or touch the x-axis?

$$R(m) = \frac{5m - 4}{m^2 - m - 2}$$

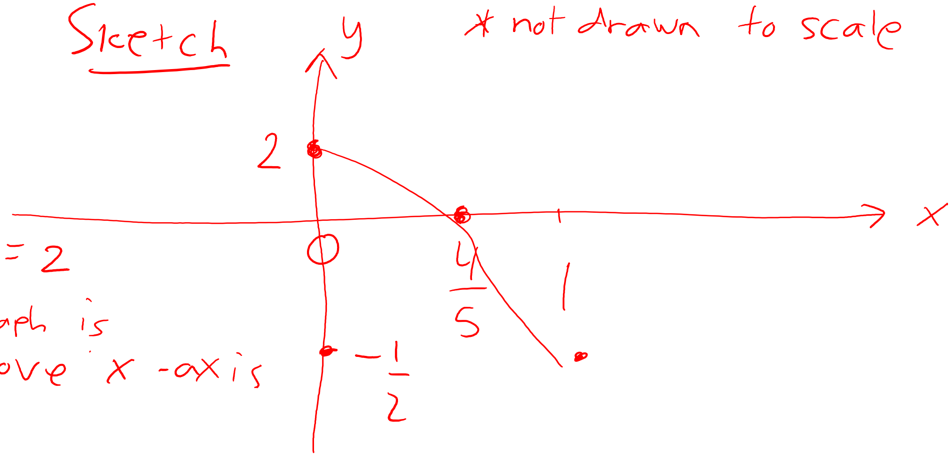
Step 4: Check

Our calculation tells us that at $x = \frac{4}{5}$, we have $y = 0$. This implies that the function at least touches the x-axis but at this point we still don't know if the function actually crosses the x-axis.

To check if the function crosses the x-axis we test points before and after $x = \frac{4}{5}$

Sketch

* not drawn to scale



When $x = 0$
$m = 0$
So $R(m) = R(0) = 2$
$\Rightarrow y = 2 \Rightarrow$ graph is above x-axis

When $x = 1$
$m = 1$
So $R(m) = R(1) = -\frac{1}{2}$
$\Rightarrow y = -\frac{1}{2} \Rightarrow$ graph is below x-axis

So the function $R(m)$ crosses x-axis at $m = 4/5$ ✓

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If there are errors or any questions, please feel free to contact me
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